MATH1520 University Mathematics for Applications **Fall 2021**

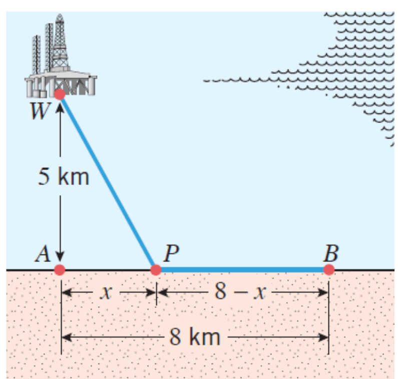
Chapter 8: Application of Derivatives III

Learning Objectives:
(1) Model and analyze optimization problems.
(2) Examine applied problems involving related rates of change.

8.1 Optimization Problems

Maximize/minimize certain quantities from application problems. This is an application of absolute extrema of functions.

Example 8.1.1. The figure shows an offshore oil well located at a point W that is 5km from the closest point A on a straight shoreline. Oil is to be piped from W to a shore point B that is 8km from A by piping it on a straight line under water from W to some shore point P between A and B and then on to B via pipe along the shoreline. If the cost of laying pipe is $\$1,000,000$/km under water and $\$500,000$/km over land, where should the point P be located to minimize the cost of laying the pipe?



Solution. Let

$$x \quad = \quad \text{distance (in kilometers) between } A \text{ and } P\text{, i.e. } |AP|$$

then,

$$|PB| = |AB| - |AP| = (8 - x) \quad \text{km}$$
$$|WP| = \sqrt{x^2 + 25} \quad \text{km}$$

Then, it follows that the total cost (in million) for the pipeline is

$$f(x) = 1(\sqrt{x^2+25}) + \frac{1}{2}(8-x) = \sqrt{x^2+25} + \frac{1}{2}(8-x), \quad x \in [0,8]. \tag{8.1}$$

f is infinitely differentiable; in particular, it is continuous on $[0,8]$. The absolute minimum exists by extreme value theorem.

$$f'(x) = \frac{x}{\sqrt{x^2+25}} - \frac{1}{2} = 0$$

Setting $f'(x) = 0$ and solving for x yields

$$\begin{aligned} \frac{x}{\sqrt{x^2+25}} &= \tfrac{1}{2} \\ x^2 &= \tfrac{1}{4}(x^2+25) \\ x &= \tfrac{5}{\sqrt{3}} \text{ or } -\tfrac{5}{\sqrt{3}} \quad \text{(rejected)} \end{aligned}$$

So, $x = \frac{5}{\sqrt{3}}$ is the only critical number in $(0,8)$.

Compare

$$f(0) = 9, \quad f(\frac{5}{\sqrt{3}}) \approx 8.330, \quad f(8) \approx 9.433.$$

The least possible cost of the pipeline (to the nearest dollar) is $\$8,330,127$, and this occurs when the point P is located at a distance of $5/\sqrt{3} \approx 2.89$ km from A.

■

Procedure to solve Optimization problem:

1. Assign variables, set up a function by expressing the quantity to be optimized in terms of the independent variable.
2. Find the absolute extrema of the function.

Example 8.1.2. Find the radius and height of the right circular cylinder of largest volume that can be inscribed in a right circular cone with radius 6 inches and height 10 inches.

Solution. Let

$$\begin{aligned} r &= \text{radius (in inches) of the cylinder} \\ h &= \text{height (in inches) of the cylinder} \\ V &= \text{volume (in cubic inches) of the cylinder} \end{aligned}$$

The formula for the volume of the inscribed cylinder is

$$V = \pi r^2 h.$$

Using similar triangles, we obtain

$$\frac{10-h}{r} = \frac{10}{6} \qquad \text{or} \qquad h = 10 - \frac{5}{3}r.$$

$$\therefore \qquad V = \pi r^2\left(10 - \frac{5}{3}r\right) = 10\pi r^2 - \frac{5}{3}\pi r^3 \tag{8.2}$$

which expresses V in terms of r alone. Because r represents a radius, it cannot be negative, and because the radius of the inscribed cylinder cannot exceed the radius of the cone, the variable r must satisfy

$$0 \le r \le 6$$

Thus, we have reduced the problem to that of finding the value (or values) of r in $[0, 6]$ for which V is maximum.
From (8.2) we obtain

$$\frac{dV}{dr} = 20\pi r - 5\pi r^2 = 5\pi r(4-r)$$

Setting $\frac{dV}{dr} = 0$ gives

$$5\pi r(4-r) = 0,$$

so $r = 0$ and $r = 4$ are critical points. Since these lie in the interval $[0, 6]$, the maximum must occur at one of the values

$$r = 0, \qquad r = 4, \qquad r = 6.$$

Substituting these values into (8.2), we have

$$V = 0, \qquad V = \frac{160}{3}\pi, \qquad V = 0$$

It tells us the maximum volume $V = \frac{160}{3}\pi$ occurs when the inscribed cylinder has radius 4 in. When $r = 4$ it follows that $h = \frac{10}{3}$. Thus, the inscribed cylinder of largest volume has radius $r = 4$ in and height $h = \frac{10}{3}$ in. ■

Example 8.1.3. Among all the rectangles with fixed area $S_0 > 0$, find the minimal perimeter.

Solution. Let one side of the rectangle has length $x > 0$ then the other side is $\frac{S_0}{x}$, and the perimeter is

$$\text{Perimeter} \quad f(x) = 2(x + \frac{S_0}{x}), \quad x \in (0, +\infty)$$

Although extreme value theorem cannot be applied on $(0, +\infty)$, we can still use the monotonicity to find the absolute extrema.

Let

$$f'(x) = 2(1 - \frac{S_0}{x^2}) = 0, \quad \Rightarrow \quad x = \sqrt{S_0} \quad \text{or} \quad -\sqrt{S_0} \text{ (rejected, not in } (0, +\infty))$$

x	$(0, \sqrt{S_0})$	$\sqrt{S_0}$	$(\sqrt{S_0}, +\infty)$
$f'(x)$	$-$	0	$+$
f	$\downarrow$	absolute min	$\uparrow$

Thus the minimal perimeter occurs when $x = \sqrt{S_0}$, i.e. it is a square. ■

8.2 Related Rates

Given rate of change of one quantity A, find the rate of change of another quantity B which is related to A. This is an application of implicit differentiation.

Example 8.2.1. A 26-foot ladder is placed against a wall. If the top of the ladder is sliding down the wall at 2 feet per second, at what rate is the bottom of the ladder moving away from the wall when the bottom of the ladder is 10 feet away from the wall?

Solution. At any time t, let

$$\begin{aligned} x(t) &= \text{the distance of the bottom of the ladder from the wall} \\ y(t) &= \text{the distance of the top of the ladder from the ground} \end{aligned}$$

x and y are related by the Pythagorean relationship:

$$x^2(t) + y^2(t) = 26^2 \tag{8.3}$$

Differentiating the above equation implicitly with respect to t, we obtain

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0. \tag{8.4}$$

The rates $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are related by equation (8.4). This is a related-rates problem.
By assumption, $\frac{dy}{dt} = -2$ (y is decreasing at a constant rate of 2 feet per second).

When $x(t) = 10$, $y(t) = \sqrt{26^2 - 10^2} = 24$ feet.

So,

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt} = \frac{-2(24)(-2)}{2(10)} = 4.8 \text{ feet per second.}$$

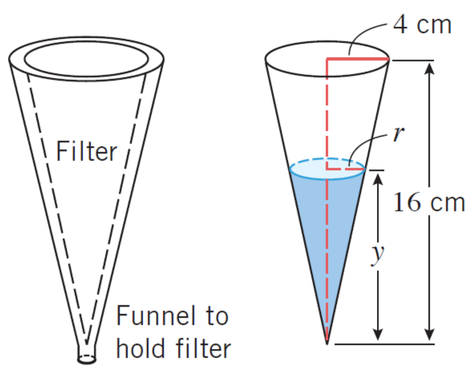
The bottom of the ladder is moving away from the wall at a rate of 4.8 feet per second. ■

Exercise 8.2.1. Again, a 26-foot ladder is placed against a wall. If the bottom of the ladder is moving away from the wall at 3 feet per second, at what rate is the top moving down when the top of the ladder is 24 feet above ground?

Procedure to Solve Related Rates Problems

1. Assign variables: independent (usually t), dependent (usually $x, y, \ldots$)
2. Find the relation between variables $x, y, \ldots$
3. Take implicit differentiation about t, solve the unkonw rate of change using the known rate of change.

Example 8.2.2. Suppose that liquid is to be cleared of sediment by allowing it to drain through a conical filter that is 16 cm high and has a radius of 4 cm at the top. Suppose also that the liquid is forced out of the cone at a constant rate of 2 cm^3/min. What is the rate of change of the depth of liquid when the liquid is 8 cm deep?



Solution. Let

$$\begin{aligned} t &= \text{time elapsed from the initial observation (min)} \\ V &= \text{volume of liquid in the cone at time } t \text{ (cm}^3\text{)} \\ y &= \text{depth of the liquid in the cone at time } t \text{ (cm)} \\ r &= \text{radius of the liquid surface at time } t \text{ (cm)} \end{aligned}$$

From the formula for the volume of a cone, the volume V, the radius r, and the depth y are related by

$$V = \frac{1}{3}\pi r^2 y \tag{8.5}$$

We are studying the related rate of V and y, so we eliminate r using similar triangle properties.

$$\frac{r}{y} = \frac{4}{16} \qquad \text{or } r = \frac{1}{4}y$$

Substituting this expression in (8.5) gives

$$V = \frac{\pi}{48}y^3$$

Differentiating both sides with respect to t we obtain

$$\frac{dV}{dt} = \frac{\pi}{48}\left(3y^2\frac{dy}{dt}\right)$$

or

$$\frac{dy}{dt} = \frac{16}{\pi y^2}\frac{dV}{dt} = \frac{16}{\pi y^2}(-2) = -\frac{32}{\pi y^2} \tag{8.6}$$

which expresses $\frac{dy}{dt}$ in terms of y.

The rate at which the depth is changing when the depth is 8 cm can be obtained from (8.6) with $y = 8$:

$$\left.\frac{dy}{dt}\right|_{y=8} = -\frac{32}{\pi(8^2)} = -\frac{1}{2\pi} \approx -0.16 \text{ cm/min}$$

So, the depth is decreasing with rate $-\frac{1}{2\pi}$ cm/min at that moment.

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